Recovery of the Collision Kernel in the
Linear Boltzmann Equation by a Finite Number of
Measurements on the Boundary

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Abstract. In this paper we consider the inverse problem of recovering the collision kernel for the time dependent linear Boltzmann equation via a finite number of boundary measurements. We prove that this kernel can be uniquely determined by at most \( k \) measurements, provided that it belongs to a finite \( k \)-dimensional vector space.

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1. Introduction

In this paper we consider an inverse problem for the linear Boltzmann equation

\[
\partial_t u + \omega \cdot \nabla_x u + qu = qK_\kappa[u] \quad \text{in} \quad (0, T) \times \mathbb{S} \times \Omega, \tag{1.1}
\]

where \( T > 0 \), \( \Omega \) is a smooth bounded convex domain of \( \mathbb{R}^N \), \( N \geq 2 \), \( \mathbb{S} \) denotes the unit sphere of \( \mathbb{R}^N \), \( q \in L^\infty(\Omega) \) and \( K_\kappa \) is the integral operator with kernel \( \kappa(x, \omega', \omega) \) defined by

\[
K_\kappa[u](t, \omega, x) = \int_\mathbb{S} \kappa(x, \omega', \omega)u(t, \omega', x) \, d\omega'. \tag{1.2}
\]

In applications, the equation (1.1) describes the dynamics of a monokinetic flow of particles in a body \( \Omega \) under the assumption that the interaction between them is negligible (which allows us to discard nonlinear terms). For instance, in the case of a low-density flux of neutrons (see [7], [10]), \( q \geq 0 \) is the total extinction coefficient and the collision kernel \( \kappa \) is given by

\[
\kappa(x, \omega', \omega) = c(x)h(x, \omega' \cdot \omega),
\]
where $c$ corresponds to the within-group scattering probability and $h$ describes the anisotropy of the scattering process. In this model, $q(x)u(t, \omega, x)$ describes the loss of particles at $x$ in the direction $\omega$ at time $t$ due to absorption or scattering and $q(x)K_\kappa[u](t, \omega, x)$ represents the production of particles at $x$ in the direction $\omega$ from those coming from directions $\omega'$.

Our focus here is the inverse problem of recovery the coefficients in (1.1) via boundary measurements. More precisely, we are interested to recover $q$ and $\kappa$ by giving the incoming flux of particles on the boundary and measuring the outgoing one. Since these operations are described mathematically by the albedo operator

$$A_{q,\kappa} : L^1(0, T; L^1(\Sigma^-; d\xi)) \longrightarrow L^1(0, T; L^1(\Sigma^+; d\xi))$$

(the spaces will be precised below), a general mathematical question concerning this inverse problem is to know if the knowledge of $A_{q,\kappa}$ uniquely determines $q, \kappa$, i.e., if the map $(q, \kappa) \mapsto A_{q,\kappa}$ is invertible.

Taking into account the applications, we have to precise this question. A first one is to know if the knowledge of $A_{q,\kappa}[f]$ for all $f$ determines $(q, \kappa)$ (infinitely many measurements); a second one is to know if the knowledge of $A_{q,\kappa}[f_j]$, for $j = 1, 2, \ldots, k$, determines $(q, \kappa)$ (finite number of measurements).

There is a wide bibliography devoted to the first problem. We specially mention the general results obtained by Choulli and Stefanov [4]: they show that $q$ and $\kappa$ are uniquely determined by the albedo operator (see also [9]). We also mention the stability results obtained by Cipolatti, Motta and Roberty (see [5] and the references therein).

There is also a lot of papers concerning the stationary case (see for instance those by V.G. Romanov [11], [12], P. Stefanov and G. Uhlmann [13], Tamasan [14], J.N. Wang [15], and also the references therein).

In this work we focus on the second question, concerning the recovery by a finite number of measurements. This may be interesting from the numerical point of view (finite element methods, for instance). Assuming that $\kappa(t, \omega, \omega) = c(x)h(\omega', \omega)$, we prove that $c$ can be uniquely determined by at most $k$ measurements, provided that $c$ belongs to a finite $k$-dimensional vector space of $C(\overline{\Omega})$. More precisely:

**Theorem 1.1:** Let $\Omega \subset \mathbb{R}^N$ be a bounded convex domain of class $C^1$, $T > \text{diam}(\Omega)$ and $\mathcal{X} = \text{span}\{\rho_1, \rho_2, \ldots, \rho_k\}$, where $\{\rho_1, \rho_2, \ldots, \rho_k\}$ is a linearly independent subset of $C(\overline{\Omega})$. We assume that $c \in \mathcal{X}$ and $\kappa(x, \omega', \omega) = c(x)h(\omega', \omega)$, where $h \in C(\mathbb{S} \times \mathbb{S})$ satisfies $h(\omega, \omega) \neq 0$ for every $\omega \in \mathbb{S}$. Then, there exist $f_1, \ldots, f_k \in C_0((0, T) \times \Sigma^-)$ and $\tilde{\omega}_1, \ldots, \tilde{\omega}_k \in \mathbb{S}$ that determine $\kappa$ uniquely.
The proof of Theorem 1.1 is based on the construction of highly oscillatory solutions (à la Calderón [1]) introduced in [5] and some arguments already used by the author in [6]. In fact, we consider solutions of the form

\[ u_j(t, \omega, x) = \chi_s(\tilde{\omega}_j, \omega) \phi_j(x - t\omega) e^{-\int_0^t \tilde{q}(x - r\omega) dr} e^{i\lambda(t-x\cdot \omega)} + R_{\lambda,s}(t, \omega, x), \]

where \( \chi_s \) converges (as \( s \to 1 \)) to \( \delta_{\tilde{\omega}_j} \), the spherical atomic measure concentrated on \( \tilde{\omega}_j \) and \( R_{\lambda,s} \) vanishes as \( \lambda \to \infty \). Therefore, by choosing \( \tilde{\omega}_j \) and \( \phi_j \) conveniently, we obtain the result.

We organize the paper as follows: in Section 2 we recall the standard functional framework in which the Cauchy problem for (1.1) is well posed in the sense of the semigroup theory and the albedo operator is defined; in Section 3, we introduce the highly oscillatory functions that will be used, in Section 4, to prove Theorem 1.1.

2. Notation and Functional Framework

In this section we introduce the notation and we recall some well known results on the Transport Operator and the semigroup it generates in the Neutronic Function Spaces (see [5] and the references therein for the proofs).

Let \( \Omega \subset \mathbb{R}^N \) \( (N \geq 2) \) be a convex and bounded domain of class \( C^1 \) and \( S \) the unit sphere of \( \mathbb{R}^N \). We denote by \( Q := S \times \Omega \) and \( \Sigma \) its boundary, i.e., \( \Sigma := S \times \partial \Omega \). For \( p \in [1, +\infty) \) we consider the space \( L^p(Q) \) with the usual norm

\[ \|u\|_{L^p(Q)} := \left( \int_Q |u(\omega, x)|^p \, dx \, d\omega \right)^{1/p}, \]

where \( d\omega \) denotes the surface measure on \( S \) associated to the Lebesgue measure in \( \mathbb{R}^{N-1} \).

For each \( u \in L^p(Q) \) we define \( A_0u \) by

\[ (A_0u)(\omega, x) := \omega \cdot \nabla_x u(\omega, x) = \sum_{k=1}^N \omega_k \frac{\partial u}{\partial x_k}(\omega, x), \quad \omega = (\omega_1, \ldots, \omega_N) \]

where the derivatives are taken in the sense of distributions in \( \Omega \).

One checks easily that setting \( \mathcal{W}_p := \{ u \in L^p(Q) ; A_0u \in L^p(Q) \} \), the operator \( (A_0, \mathcal{W}_p) \) is a closed densely defined operator and \( \mathcal{W}_p \) with the graph norm is a Banach space.
For every \( \sigma \in \partial \Omega \), we denote \( \nu(\sigma) \) the unit outward normal at \( \sigma \in \partial \Omega \) and we consider the sets (respectively, the incoming and outgoing boundaries)

\[
\Sigma^\pm := \{(\omega, \sigma) \in \mathbb{S} \times \partial \Omega; \pm \omega \cdot \nu(\sigma) > 0\}.
\]

In order to well define the albedo operator as a trace operator on the outgoing boundary, we consider \( L^p(\Sigma^\pm; d\xi) \), where \( d\xi := |\omega \cdot \nu(\sigma)| d\sigma d\omega \), and we introduce the spaces

\[
\tilde{W}_p^\pm := \{ u \in W_p; u|_{\Sigma^\pm} \in L^p(\Sigma^\pm; \xi) \},
\]

which are Banach spaces if equipped with the norms

\[
\|u\|_{\tilde{W}_p^\pm} := \left( \|u\|_{W_p}^p + \int_{\Sigma^\pm} |\omega \cdot \nu(\sigma)||u(\omega, \sigma)|^p d\sigma d\omega \right)^{1/p}.
\]

The next two lemmas concern the continuity and surjectivity of the trace operators (see [2], [3] and [5]):

\[
\gamma^\pm : \tilde{W}_p^\pm \to L^p(\Sigma^\mp; d\xi), \quad \gamma^\pm(u) := u|_{\Sigma^\mp}.
\]

**Lemma 2.1:** Let \( 1 \leq p < +\infty \). Then there exists \( C > 0 \) (depending only on \( p \)) such that

\[
\int_{\Sigma^\mp} |\omega \cdot \nu(\sigma)||u(\omega, \sigma)|^p d\sigma d\omega \leq C \|u\|_{\tilde{W}_p^\pm}^p, \quad \forall u \in \tilde{W}_p^\pm.
\]

Moreover, if \( p > 1 \) and \( 1/p + 1/p' = 1 \), we have the Gauss identity

\[
\int_Q \text{div}_x(uv \omega) dx d\omega = \int_{\Sigma} \omega \cdot \nu(\sigma) u(\omega, \sigma)v(\omega, \sigma) d\sigma d\omega,
\]

for all \( u \in \tilde{W}_p^\pm \) and \( v \in \tilde{W}_{p'}^\pm \).

As an immediate consequence of Lemma 2.1, we can introduce the space

\[
\tilde{W}_p^\pm := \{ f \in W_p; \int_{\Sigma} |\omega \cdot \nu(\sigma)||f(\omega, \sigma)|^p d\omega d\sigma < +\infty \}
\]

and we have that \( \tilde{W}_p^+ = \tilde{W}_p^- = \tilde{W}_p \) with equivalent norms.

**Lemma 2.2:** The trace operators \( \gamma^\pm \) are surjective from \( \tilde{W}_p^\pm \) onto \( L^p(\Sigma^\mp; d\xi) \). More precisely, for each \( f \in L^p(\Sigma^\mp; d\xi) \), there exists \( h \in \tilde{W}_p^\pm \) such that \( \gamma^\pm(h) = f \) and

\[
\|h\|_{\tilde{W}_p^\pm} \leq C \|f\|_{L^p(\Sigma^\mp, d\xi)},
\]

where \( C > 0 \) is independent of \( f \).

We consider the operator \( A : D(A) \to L^p(Q) \), defined by \( (Au)(\omega, x) := \omega \cdot \nabla u(\omega, x) \), with \( D(A) := \{ u \in \tilde{W}_p; \gamma^-(u) = 0 \} \).
Theorem 2.3: The operator $A$ is $m$-accretive in $L^p(Q)$, for $p \in [1, +\infty)$.

Corollary 2.4: Let $f \in L^p(Q)$, $p \in [1, +\infty)$ and assume that $u \in D(A)$ is a solution of $u + Au = f$. If $f \geq 0$ a.e. in $Q$, then $u \geq 0$ a.e. in $Q$. In particular, it follows that

$$
\|u\|_{L^1(Q)} \leq \|f\|_{L^1(Q)}.
$$

It follows from Theorem 2.3 and Corollary 2.4 that the operator $A$ generates a positive semigroup $\{U_0(t)\}_{t \geq 0}$ of contractions acting on $L^p(Q)$.

Let $q \in L^\infty(\Omega)$ and $\kappa : \Omega \times S \times S \to \mathbb{R}$ be a real measurable function satisfying

$$
\begin{aligned}
\left\{ \begin{array}{l}
\int_S |\kappa(x, \omega, \omega')| \, d\omega' \leq M_1 \text{ a.e. } \Omega \times S,
\int_S |\kappa(x, \omega, \omega')| \, d\omega \leq M_2 \text{ a.e. } \Omega \times S.
\end{array} \right.
\end{aligned}
$$

(2.4)

Associated to these functions, we define the following continuous operators:

1) $B \in \mathcal{L}(L^p(Q), L^p(Q))$ defined by $B[u](\omega, x) = q(x)u(\omega, x)$,

2) $K_\kappa[u](\omega, x) = \int_S \kappa(x, \omega, \omega')u(\omega', x) \, d\omega'$.

It follows from (2.4) that $K_\kappa \in \mathcal{L}(L^p(Q), L^p(Q)) \quad \forall p \in [1, +\infty)$ and (see [7])

$$
\|K_\kappa[u]\|_{L^p(Q)} \leq M_1^{1/p'}M_2^{1/p}\|u\|_{L^p(Q)} \leq \max\{M_1, M_2\}\|u\|_{L^p(Q)}.
$$

(2.5)

The operator $A + B - K_\kappa : D(A) \to L^p(Q)$ generates a $c_0$-semigroup $\{U(t)\}_{t \geq 0}$ on $L^p(Q)$ satisfying

$$
\|U(t)\|_{L^\infty} \leq e^{Ct}, \quad C := \|q^-\|_{\infty} + M_2.
$$

(2.6)

We consider the initial-boundary value problem for the linear Boltzmann equation

$$
\begin{aligned}
\partial_t u(t, \omega, x) + \omega \cdot \nabla u(t, \omega, x) + q(x)u(t, \omega, x) = q_{K_\kappa}[u](t, \omega, x)
\end{aligned}
$$

(2.7)

\begin{align*}
& u(0, \omega, x) = 0, \quad (\omega, x) \in S \times \Omega \\
& u(t, \omega, \sigma) = f(t, \omega, \sigma), \quad (\omega, \sigma) \in \Sigma^-, \quad t \in (0, T),
\end{align*}

where $q \in L^\infty(\Omega)$, $K_\kappa[u]$ is defined by (1.2) with $\kappa$ satisfying (2.4).

By the previous results, it follows that, for $f \in L^p(0, T; L^p(\Sigma^-, d\xi))$, $p \in [1, +\infty)$, there exists a unique solution $u \in C([0, T]; \mathcal{W}_p) \cap C^1([0, T]; L^p(Q))$ of (2.7). This solution $u$ allows us to define the albedo operator

$$
\begin{aligned}
A_{q, \kappa} : L^p(0, T; L^p(\Sigma^-, d\xi)) \to L^p(0, T; L^p(\Sigma^+, d\xi))
\end{aligned}
$$

$$
A_{q, \kappa}[f](t, \omega, \sigma) := u(t, \omega, \sigma), \quad (\omega, \sigma) \in \Sigma^+.
$$

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As a consequence of Lemmas 2.1 and 2.2, $A_{q,\kappa}$ is a linear and bounded operator.

We also consider the following backward-boundary value problem, called the adjoint problem of (2.7):

\[
\begin{cases}
\partial_t u^*(t, \omega, x) + \omega \cdot \nabla u^*(t, \omega, x) - q(x)u^*(t, \omega, x) = -qK^*_\kappa[u^*](t, \omega, x) \\
u^*(T, \omega, x) = 0, \quad (\omega, x) \in S \times \Omega \\
u^*(t, \omega, \sigma) = f^*(t, \omega, \sigma), \quad (\omega, \sigma) \in \Sigma^+, \ t \in (0, T),
\end{cases}
\]

where $f^* \in L^{p'}(0, T; L^{p'}(\Sigma^+, d\xi))$, $p' \in [1, +\infty)$,

\[K^*_\kappa[u^*](t, \omega', x) := \int_S \kappa(x, \omega', \omega)u^*(t, \omega, x) \, d\omega\]

with the corresponding albedo operator $A^*_{q,\kappa}$

\[A^*_{q,\kappa} : L^{p'}(0, T; L^{p'}(\Sigma^+, d\xi)) \to L^{p'}(0, T; L^{p'}(\Sigma^-, d\xi))\]

\[A^*_{q,\kappa}[f^*](t, \omega, \sigma) := u^*(t, \omega, \sigma), \quad (\omega, \sigma) \in \Sigma^-\]

The operators $A_{q,\kappa}$ and $A^*_{q,\kappa}$ satisfy the following property:

**Lemma 2.5:** Let $f \in L^p(0, T; L^p(\Sigma^-; d\xi))$ and $f^* \in L^{p'}(0, T; L^{p'}(\Sigma^+; d\xi))$, where $p, p' \in (1, +\infty)$ are such that $1/p + 1/p' = 1$. Then, we have

\[
\int_0^T \int_{\Sigma^+} (\omega \cdot \nu(\sigma))f(t, \omega, \sigma)A^*_{q,\kappa}[f^*](t, \omega, \sigma) \, d\sigma \, dt =
\]

\[
= -\int_0^T \int_{\Sigma^+} (\omega \cdot \nu(\sigma))f^*(t, \omega, \sigma)A_{q,\kappa}[f](t, \omega, \sigma) \, d\sigma \, dt.
\]

**Proof:** It is a direct consequence of Lemma 2.1. Let $u(t, \omega, x)$ the solution of (2.7) with boundary condition $f$ and $u^*(t, \omega, x)$ the solution of (2.8) with boundary $f^*$. We obtain the result by using (2.3), once the equation in (2.7) is multiplied by $u^*$ and integrated over $(0, T) \times Q$. \qed

As a direct consequence of Lemma 2.5, we have:

**Lemma 2.6:** Let $T > 0$, $q_1, q_2 \in L^\infty(\Omega)$ and $\kappa_1, \kappa_2$ satisfying (2.4). Assume that $u_1$ is the solution of (2.7) with coefficients $q_1, \kappa_1$ and satisfying the boundary condition $f \in L^p(0, T; L^p(\Sigma^-, d\xi))$, $p \in (1, +\infty)$ and that $u_2^*$ is the solution of (2.8), with $q_2, \kappa_2$
and boundary condition $f^* \in L^{p'} (0, T; L^{p'}(\Sigma^+, d\xi))$, $1/p + 1/p' = 1$. Then we have

$$\int_0^T \int_Q (q_2(x) - q_1(x)) u_1(t, \omega, x) u_2^*(t, \omega, x) \, dx \, d\omega \, dt$$

$$- \int_0^T \int_Q (q_2(x) K_{\kappa_2} [u_1](t, \omega, x) - q_1(x) K_{\kappa_1} [u_1](t, \omega, x)) u_2^*(t, \omega, x) \, dx \, d\omega \, dt$$

$$= \int_0^T \int_{\Sigma^+} (\omega \cdot \nu(\sigma)) [A_{q_1, \kappa_1} [f] - A_{q_2, \kappa_2} [f]] (t, \omega, \sigma) f^*(t, \omega, \sigma) \, d\sigma \, d\omega \, dt.$$

### 3. Highly Oscillatory Solutions

In this section we prove some technical results related to special solutions of (2.7) and (2.8) that will be useful in the proof of Theorem 1.1. We denote by $\tilde{q}$ the zero extension of $q$ in the exterior of $\Omega$.

**Proposition 3.1:** Let $T > 0$, $q_1, q_2 \in L^\infty(\Omega)$, and $\kappa$ satisfying (2.4). We consider $\psi_1, \psi_2 \in C(S, C_0^\infty(\mathbb{R}^N))$ such that

$$\text{supp } \psi_1(\omega, \cdot) \cap \overline{\Omega} = (\text{supp } \psi_2(\omega, \cdot) + T\omega) \cap \overline{\Omega} = \emptyset, \quad \forall \omega \in S. \quad (3.1)$$

Then, there exists $C_0 > 0$ such that, for each $\lambda > 0$, there exist $R_{1, \lambda} \in C([0, T]; \tilde{W}_2)$ and $R_{2, \lambda}^* \in C([0, T]; \tilde{W}_2)$ satisfying

$$\|R_{1, \lambda}\|_{C([0, T]; L^2(Q))} \leq C_0, \quad \|R_{2, \lambda}^*\|_{C([0, T]; L^2(Q))} \leq C_0, \quad (3.2)$$

for which the functions $u_1, u_2^*$ defined by

$$\begin{cases}
    u_1(t, \omega, x) := \psi_1(\omega, x - t\omega) e^{-\int_0^t \tilde{q}_1(x - s\omega) \, ds} e^{i\lambda(t - \omega \cdot x)} + R_{1, \lambda}(t, \omega, x) \\
    u_2^*(t, \omega, x) := \psi_2(\omega, x - t\omega) e^{\int_0^t \tilde{q}_2(x - s\omega) \, ds} e^{-i\lambda(t - \omega \cdot x)} + R_{2, \lambda}^*(t, \omega, x)
\end{cases} \quad (3.3)$$

are solutions of (2.7) with $q = q_1$ and (2.8) with $q = q_2$ respectively. Moreover, if $\kappa \in L^\infty(\Omega; L^2(S \times S))$, then we have

$$\lim_{\lambda \to +\infty} \|R_{1, \lambda}\|_{C([0, T]; L^2(Q))} = \lim_{\lambda \to +\infty} \|R_{2, \lambda}^*\|_{C([0, T]; L^2(Q))} = 0. \quad (3.4)$$
Proof: Let $u$ be the function

$$u(t, \omega, x) := \psi_1(\omega, x - t\omega)e^{-\int_0^t \tilde{q}_1(x-s\omega) \, ds} e^{i\lambda(t-\omega \cdot x)} + R(t, \omega, x).$$

(3.5)

By direct calculations, we easily verify that

$$\partial_t u + \omega \cdot \nabla u + q_1 u - q_1 K_\kappa[u] = \partial_t R + \omega \cdot \nabla R + q_1 R - q_1 K_\kappa[R] - e^{i\lambda t} q_1 Z_{1, \lambda},$$

where

$$Z_{1, \lambda}(t, \omega, x) := \int_{\Sigma} \kappa(x, \omega', \omega)\psi_1(\omega', x - t\omega') e^{-\int_0^t \tilde{q}_1(x-s\omega') \, ds} e^{-i\lambda \omega' \cdot x} d\omega'.$$

(3.6)

From (2.6), there exists $R_{1, \lambda} \in C^1([0, T]; L^2(Q)) \cap C([0, T]; D(A))$ a unique solution of

$$\begin{cases}
\partial_t R + \omega \cdot \nabla R + q_1 R = q_1 K_\kappa[R] + e^{i\lambda t} q_1 Z_{1, \lambda}, \\
R(0, \omega, x) = 0, \quad (\omega, x) \in S \times \Omega, \\
R(t, \omega, \sigma) = 0, \quad (\omega, \sigma) \in \Sigma^-,
\end{cases}$$

(3.7)

and it follows from (3.1) that the function $u$ defined by (3.5) satisfies (2.7) with boundary condition

$$f_\lambda(t, \omega, \sigma) := \psi_1(\omega, \sigma - t\omega)e^{-\int_0^t \tilde{q}_1(\sigma-s\omega) \, ds} e^{i\lambda(t-\omega \cdot \sigma)}, \quad (\omega, \sigma) \in \Sigma^-.$$

Multiplying both sides of the equation in (3.7) by the complex conjugate of $R$, integrating it over $Q$ and taking its real part, we get, from Lemma 2.1,

$$\frac{1}{2} \frac{d}{dt} \int_Q |R(t)|^2 d\omega dx + \frac{1}{2} \int_{\Sigma^+} \omega \cdot \nu(\sigma)|R(t)|^2 d\omega d\sigma + \int_Q q_1 |R(t)|^2 d\omega dx - \Re \int_Q q_1 K_\kappa[R](t) \overline{R(t)} d\omega dx = \Re \left[ e^{i\lambda t} \int_Q q_1 Z_{1, \lambda}(t) \overline{R(t)} d\omega dx \right].$$

It follows from the Cauchy-Schwarz inequality and (2.5) that

$$\int_Q |K_\kappa[R(t)]| |R(t)|^2 d\omega dx \leq C_1\|R(t)\|_{L^2(Q)}^2,$$

where $C_1 := \max\{M_1, M_2\}$. Therefore, we obtain

$$\frac{d}{dt}\|R(t)\|_{L^2(Q)}^2 \leq C_2\|q_1\|_{\infty}\|R(t)\|_{L^2(Q)}^2 + \|q_1\|_{\infty}\|Z_{1, \lambda}(t)\|_{L^2(Q)}^2,$$

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where $C_2 := 3 + 2C_1$. Since $R(0) = 0$, we get, by integrating this last inequality on $[0, t]$,
\[
\|R(t)\|_{L^2(Q)}^2 \leq \|q_1\|_\infty e^{\|q_1\|_\infty TC_2} \int_0^t \|Z_{1,\lambda}(\tau)\|_{L^2(Q)}^2 \, d\tau, \quad \forall t \in [0, T]. \tag{3.8}
\]

The first inequality in (3.2) follows easily because $|Z_{1,\lambda}(t, \omega, x)| \leq \|\psi_1\|_\infty e^{\|q_1\|_\infty T} M_1$ and, as the same arguments hold for $u_2^* \lambda$ and $R_{2,\lambda}$, we also obtain the second inequality.

We assume now $\kappa \in L^\infty(\Omega; L^2(S \times S))$. For each $x \in \mathbb{R}^N$, the map $\omega' \mapsto \exp(i\lambda\omega' \cdot x)$ converges weakly to zero in $L^2(S)$ when $\lambda \to +\infty$ and the integral operator with kernel $\kappa(x, \cdot, \cdot)$ is compact in $L^2(S)$. So, we obtain from (3.6),
\[
\lim_{\lambda \to +\infty} \|Z_{1,\lambda}(t, \cdot, x)\|_{L^2(S)} = 0 \quad \text{a.e. in } [0, T] \times \Omega.
\]
Moreover, $\|Z_{1,\lambda}(t, \cdot, x)\|_{L^2(S)} \leq C$, where $C > 0$ is a constant that does not depend on $\lambda$. The Lebesgue’s Dominated Convergence Theorem implies that
\[
\lim_{\lambda \to +\infty} \|Z_{1,\lambda}\|_{L^2([0,T] \times Q)} = 0. \tag{3.9}
\]
From (3.9) and (3.8) we obtain (3.4), and our proof is complete. $\square$

**Corollary 3.2**: Under the hypothesis of Proposition 3.1, if $q_1, q_2 \in C(\overline{\Omega})$ and $\kappa \in L^\infty(\Omega; C(S \times S))$, we have, for every $\omega \in S$,
\[
\lim_{\lambda \to +\infty} \|R_{1,\lambda}(\cdot, \omega, \cdot)\|_{C([0,T];L^2(\Omega))} = \lim_{\lambda \to +\infty} \|R_{2,\lambda}^*(\cdot, \omega, \cdot)\|_{C([0,T];L^2(\Omega))} = 0.
\]

**Proof**: By multiplying both sides of the equation in (3.7) by the complex conjugate of $R(t, \omega, x)$, integrating it over $\Omega$, taking its real part and applying the Hölder inequality, we get
\[
\frac{d}{dt} \|R(t, \omega)\|_{L^2(\Omega)}^2 \leq 4 \|q_1\|_\infty \|R(t, \omega)\|_{L^2(\Omega)}^2 \]
\[
+ \|q_1\|_\infty \left( \|K_\kappa[R](t, \omega)\|_{L^2(\Omega)}^2 + \|Z_{1,\lambda}(t, \omega)\|_{L^2(\Omega)}^2 \right). \tag{3.10}
\]

Since
\[
|K_\kappa[R](t, \omega, x)| \leq \int_S |\kappa(x, \omega', \omega)||R(t, \omega', x)| \, d\omega'
\]
\[
\leq \left( \int_S |\kappa(x, \omega', \omega)| \, d\omega' \right)^{1/2} \left( \int_S |\kappa(x, \omega', \omega)||R(t, \omega', x)|^2 \, d\omega' \right)^{1/2}
\]
\[
\leq M_1^{1/2} \|\kappa\|_\infty^{1/2} \left( \int_S |R(t, \omega', x)|^2 \, d\omega' \right)^{1/2},
\]
we obtain
\[
\|K_\kappa[R](t, \omega)\|_{L^2(\Omega)}^2 \leq M_1 \|\kappa\|_\infty \|R(t)\|_{L^2(Q)}^2. \tag{3.11}
\]

From (3.8), (3.10) and (3.11) we have
\[
\frac{d}{dt}\|R(t, \omega)\|_{L^2(\Omega)}^2 \leq 4\|q_1\|_\infty \|R(t, \omega)\|_{L^2(\Omega)}^2 + C \left(\|Z_{1, \lambda}\|_{L^2((0, T) \times \Omega)}^2 + \|Z_{1, \lambda}(t, \omega)\|_{L^2(\Omega)}^2\right).
\]

Now, integrating this last inequality on time, we get
\[
\|R(t, \omega)\|_{L^2(\Omega)}^2 \leq C e^{\|q_1\|_\infty T} \left(t\|Z_{1, \lambda}\|_{L^2((0, T) \times \Omega)}^2 + \int_0^t \|Z_{1, \lambda}(\tau, \omega)\|_{L^2(\Omega)}^2 d\tau\right)
\leq C e^{\|q_1\|_\infty T} \left(T\|Z_{1, \lambda}\|_{L^2((0, T) \times \Omega)}^2 + \|Z_{1, \lambda}(\cdot, \omega, \cdot)\|_{L^2((0, T) \times \Omega)}^2\right).
\]

From Proposition 3.1 we know that \(\|Z_{1, \lambda}\|_{L^2((0, T) \times \Omega)} \to 0\) as \(\lambda \to +\infty\). On the other hand, as the map \(\omega' \mapsto e^{i\omega' \cdot x}\) converges weakly to zero in \(L^2(\mathbb{S})\), we have from (3.6), for almost \(x \in \Omega\),
\[
\lim_{\lambda \to -\infty} Z_{1, \lambda}(t, \omega, x) = 0, \quad \forall \omega \in S, \forall t \in [0, T]
\]
and the conclusion follows from the Lebesgue’s Theorem. \(\square\)

**Lemma 3.3:** We assume that \(q \in L^\infty(\Omega)\) and \(\kappa\) satisfies (2.4). Let \(S^*_\lambda\) be the solution of
\[
\begin{cases}
\partial_t S + \omega \cdot \nabla S - q S = -q K^*_\kappa[S] + q e^{-i\lambda t} Z, \\
S(T, \omega, x) = 0, \quad (\omega, x) \in S \times \Omega, \\
S(t, \omega, \sigma) = 0, \quad (\omega, \sigma) \in \Sigma^+,
\end{cases}
\tag{3.12}
\]
where \(Z \in H^1(0, T; L^2(Q))\) such that \(Z(T) = 0\). Then we have
\[
\|S^*_\lambda\|_{C([0, T]; L^2(Q))} \leq C_0 \quad \text{and} \quad \lim_{\lambda \to -\infty} \|S^*_\lambda\|_{H^{-1}(0, T; L^2(Q))} = 0, \tag{3.13}
\]
where \(C_0\) is a constant independent of \(\lambda\).

**Proof:** Multiplying both sides of the equation in (3.12) by the complex conjugate of \(S^*_\lambda\), integrating it over \(Q\) and taking its real part, we get
\[
\frac{1}{2} \frac{d}{dt}\|S^*_\lambda(t)\|_{L^2(Q)}^2 + \frac{1}{2} \int_{\Sigma^+} (\omega \cdot \nu(\sigma)) |S^*_\lambda(t, \omega, \sigma)| d\omega d\sigma \geq -\|q\|_\infty \|S^*_\lambda(t)\|_{L^2(Q)}^2
\]
\[
- \|q\|_\infty \|K^*_\kappa[S](t)\|_{L^2(Q)} \|S^*_\lambda(t)\|_{L^2(Q)} - \|q\|_\infty \|Z(t)\|_{L^2(Q)} \|S^*_\lambda(t)\|_{L^2(Q)}
\]

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Since \( \|K^*_\kappa[S](t)\|_{L^2(Q)} \leq \max\{M_1, M_2\}\|S^*_\lambda(t)\|_{L^2(Q)} \), we have

\[
\frac{d}{dt}\|S^*_\lambda(t)\|_{L^2(Q)}^2 \geq -C_2\|S^*_\lambda(t)\|_{L^2(Q)}^2 - \|q\|_\infty\|Z(t)\|_{L^2(Q)}^2,
\]

where \( C_2 = (3 + 2 \max\{M_1, M_2\})\|q\|_\infty \). Integrating this last inequality on \([t, T]\) and taking into account that \( S^*_\lambda(T) = 0 \), we obtain

\[
\|S^*_\lambda(t)\|_{L^2(Q)}^2 \leq \|q\|_\infty e^{C_2(T-t)} \int_t^T \|Z(\tau)\|_{L^2(Q)}^2 d\tau \leq \|q\|_\infty e^{C_2T} \|Z\|_{L^2(0,T;L^2(Q))}^2
\]

and the inequality in (3.13) follows easily.

We consider now

\[
w_\lambda(t, \omega, x) := \int_t^T S^*_\lambda(\tau, \omega, x) d\tau, \quad h(t, \omega, x) := \int_t^T e^{-i\lambda\tau} Z(\tau, \omega, x) d\tau.
\]

Then, it is easy to check that \( w_\lambda \) satisfies

\[
\begin{aligned}
\partial_\tau w + \omega \cdot \nabla w - q_2 w &= -qK^*_\kappa[w] + qh, \\
w(T, \omega, x) &= 0, \quad (\omega, x) \in \mathbb{S} \times \Omega, \\
w(t, \omega, \sigma) &= 0, \quad (\omega, \sigma) \in \Sigma^+,
\end{aligned}
\]

Multiplying both sides of the equation in (3.16) by the complex conjugate of \( w_\lambda \), integrating it over \( Q \), taking its real part and applying the Cauchy-Schwarz inequality, we get as before,

\[
\|w_\lambda(t)\|_{L^2(Q)}^2 \leq \|q\|_\infty e^{C_2(T-t)} \|h\|_{L^2(0,T;L^2(Q))}^2 \leq \|q\|_\infty T^2 e^{C_2T} \|Z\|_{L^2(0,T;L^2(Q))}^2.
\]

As \( S^*_\lambda = -\partial_\tau w_\lambda \), it follows from (3.14) and (3.17) that the set \( \{w_\lambda\} \) is bounded in \( C^1([0,T];L^2(Q)) \) and, in particular, is relatively compact in \( C([0,T];L^2(Q)) \).

On the other hand, by integrating by parts the second integral in (3.15), it is easy to check that there exists \( C > 0 \) (depending only on \( T \)) such that

\[
\|h\|_{L^2(0,T;L^2(Q))} \leq \frac{C}{\lambda} \|Z\|_{H^1(0,T;L^2(Q))}.
\]

Hence, by (3.17), it follows that \( \|w_\lambda\|_{C([0,T];L^2(Q))} \to 0 \) as \( \lambda \to \infty \). Since the partial derivative in \( t \), \( \partial_t : C([0,T];L^2(Q)) \to H^{-1}(0,T;L^2(Q)) \), is a continuous operator, there exists a constant \( C_3 > 0 \) such that

\[
\|S^*_\lambda\|_{H^{-1}(0,T;L^2(Q))} = \|\partial_t w_\lambda\|_{H^{-1}(0,T;L^2(Q))} \leq C_2 \|w_\lambda\|_{C(0,T;L^2(Q))}
\]

and we have the conclusion. \( \square \)
4. Recovery by a Finite Number of Boundary Measurements

In this section we assume that \( \{\rho_1, \rho_2, \ldots, \rho_k\} \) is a given linearly independent set of functions of \( C(\overline{\Omega}) \) and we denote \( X := \text{span}\{\rho_1, \rho_2, \ldots, \rho_k\} \). For each \( \tilde{\omega} \in S \) we consider \( P_{\tilde{\omega}}[\rho_i] \) the X-ray transform of \( \rho_i \) in the direction \( \tilde{\omega} \), i.e.,

\[
P_{\tilde{\omega}}[\rho_i](x) = \int_{-\infty}^{\infty} \rho_i(x + t\tilde{\omega}) \, dt
\]

and, for each \( \varepsilon > 0 \), \( \Omega_{\varepsilon} := \{ x \in \mathbb{R}^N \setminus \overline{\Omega} ; \text{dist}(x, \Omega) < \varepsilon \} \).

The following Lemma, which the proof is given in [6], will be essential for the proof of Theorem 1.1:

**Lemma 4.1:** For all \( \varepsilon > 0 \), there exist \( \tilde{\omega}_j \in S \) and \( \phi_j \in C^\infty_0(\Omega_{\varepsilon}) \), \( j = 1, \ldots, k \), such that the matrix \( A = (a_{ij}) \), with entries defined by

\[
a_{ij} := \int_{\mathbb{R}^N} P_{\tilde{\omega}_j}[\rho_i](x) \phi_j^2(x) \, dx,
\]

is invertible.

In order to prove Theorem 1.1, we define, for \( 0 < r < 1 \), the function \( \chi_r : S \times S \to \mathbb{R} \) as \( \chi_r(\tilde{\omega}, \omega) := P(r\tilde{\omega}, \omega) \), where \( P \) is the Poisson kernel for \( B_1(0) \), i.e.,

\[
P(x, y) := \frac{1 - |x|^2}{\alpha_N |x - y|^N}.
\]

From the well known properties of \( P \) (see [8]), we have

\[
\int_S \chi_r(\tilde{\omega}, \omega) \, d\omega = 1, \quad \forall r \in (0, 1), \quad \forall \tilde{\omega} \in S,
\]

\[
\lim_{r \to 1} \int_S \chi_r(\tilde{\omega}, \omega) \psi(\omega) \, d\omega = \psi(\tilde{\omega}), \tag{4.2}
\]

where the above limit is taken in the topology of \( L^p(S) \), \( p \in [1, +\infty) \) and uniformly on \( S \) if \( \psi \in C(S) \). We are now in position to prove our main result.

**Proof of Theorem 1.1:** Let \( \varepsilon := (T - \text{diam}(\Omega))/2 \). We assume that \( q_1 = q_2 = q \) and \( \kappa_i(x, \omega', \omega) = c_i(x)h(\omega', \omega) \), where \( c_1, c_2 \in X \). For \( \tilde{\omega} \in S \), we define \( \psi_1(\omega, x) = \chi_s(\tilde{\omega}, \omega)\phi(x) \) and \( \psi_2(\omega, x) = \chi_r(\tilde{\omega}, \omega)\phi(x) \), where \( 0 < r < s < 1 \) and \( \phi \in C^\infty_0(\Omega_{\varepsilon}) \). Then
ψ_1 and ψ_2 satisfy the condition (3.1) and we may consider the solutions u_1 and u^*_2 defined by (3.3), i.e.,

\[ u_1(t, \omega, x) = \chi_s(\tilde{\omega}, \omega)\phi(x - t\omega)e^{-\int_0^t \tilde{q}(x - \tau\omega)d\tau}e^{i\lambda(t-x\cdot \omega)} + R_{1, \lambda, s}(t, \omega, x), \]
\[ u^*_2(t, \omega, x) = \chi_r(\tilde{\omega}, \omega)\phi(x - t\omega)e^{\int_0^t \tilde{q}(x - \tau\omega)d\tau}e^{-i\lambda(t-x\cdot \omega)} + R^*_{2, \lambda, r}(t, \omega, x), \]

where λ > 0 will be chosen a posteriori. We shall write

\[ \Phi_{\lambda}(t, \omega, x) := \phi(x - t\omega)e^{-\int_0^t \tilde{q}(x - \tau\omega)d\tau}e^{i\lambda(t-x\cdot \omega)} \]
\[ \Psi_{\lambda}(t, \omega, x) := \phi(x - t\omega)e^{\int_0^t \tilde{q}(x - \tau\omega)d\tau}e^{-i\lambda(t-x\cdot \omega)} \]

in such a way that

\[ u_1(t, \omega, x) = \chi_s(\tilde{\omega}, \omega)\Phi_{\lambda}(t, \omega, x) + R_{1, \lambda, s}(t, \omega, x), \]
\[ u^*_2(t, \omega, x) = \chi_r(\tilde{\omega}, \omega)\Psi_{\lambda}(t, \omega, x) + R^*_{2, \lambda, r}(t, \omega, x). \]

Substituting u_1 and u^*_2 in the identity given in Lemma 2.6, we have

\[ J(\lambda, r, s) = L(\lambda, r, s), \] (4.4)

where

\[ J(\lambda, r, s) := \int_0^T \int_{\Omega} q(x)(c_1(x) - c_2(x))K_h[u_1](t, \omega, x)u^*_2(t, \omega, x) \, dx d\omega dt, \]
\[ L(\lambda, r, s) := \int_0^T \int_{\Sigma^+} (\omega \cdot \nu(\sigma))(A_1[f_{\lambda, s}] - A_2[f_{\lambda, s}])f^*_{\lambda, r} \, d\sigma d\omega dt. \]

In the above formulas, we are denoting \( A_i = A_{c_i}, i = 1, 2 \) and

\[ f_{\lambda, s}(t, \omega, \sigma) := \chi_s(\tilde{\omega}, \sigma)\Phi_{\lambda}(t, \omega, \sigma), \quad (\omega, \sigma) \in \Sigma^-, \]
\[ f^*_{\lambda, r}(t, \omega, \sigma) := \chi_r(\tilde{\omega}, \sigma)\Psi_{\lambda}(t, \omega, \sigma), \quad (\omega, \sigma) \in \Sigma^+. \] (4.5)

In particular, it follows from the definition of the Albedo Operator and (4.3),

\[ A_1[f_{\lambda, s}] - A_2[f_{\lambda, s}] = R_{1, \lambda, s} - R^*_{2, \lambda, s}, \quad \text{on} \; (0, T) \times \Sigma^+. \] (4.6)
By denoting \( \eta(x) = \tilde{q}(x)(\tilde{c}_1(x) - \tilde{c}_2(x)) \) and by considering the special form of \( u_1 \) and \( u_2^* \), we may write \( J(\lambda, r, s) \) as \( J = J_1 + J_2 + J_3 + J_4 \), where

\[
J_1(\lambda, r, s) = \int_0^T \int_Q \eta(x) \int_S h(\omega', \omega) \chi_s(\tilde{\omega}, \omega') \Phi_\lambda(t, \omega', x) d\omega' \times \chi_r(\tilde{\omega}, \omega) \Psi_\lambda(t, \omega, x) dx d\omega dt,
\]

\[
J_2(\lambda, r, s) = \int_0^T \int_Q \eta(x) \int_S h(\omega', \omega) \chi_s(\tilde{\omega}, \omega') \Phi_\lambda(t, \omega', x) d\omega' \times R_{2, \lambda, r}^*(t, \omega, x) dx d\omega dt,
\]

\[
J_3(\lambda, r, s) = \int_0^T \int_Q \eta(x) \int_S h(\omega', \omega) R_{1, \lambda, s}(t, \omega', x) d\omega' \times \chi_r(\tilde{\omega}, \omega) \Psi_\lambda(t, \omega, x) dx d\omega dt,
\]

\[
J_4(\lambda, r, s) = \int_0^T \int_Q \eta(x) \int_S h(\omega', \omega) R_{1, \lambda, s}(t, \omega', x) d\omega' \times R_{2, \lambda, r}^*(t, \omega, x) dx d\omega dt.
\]

Taking the limit as \( r \to 1^- \) in the above expressions, we get from the definition of \( \chi_r \), \( J_i(\lambda, r, s) \to J_i(\lambda, s) \), where

\[
J_1(\lambda, s) = \int_0^T \int_\Omega \eta(x) \int_S h(\omega', \omega) \chi_s(\tilde{\omega}, \omega') \Phi_\lambda(t, \omega', x) d\omega' \times \Psi_\lambda(t, \tilde{\omega}, x) dx dt,
\]

\[
J_2(\lambda, s) = \int_0^T \int_\Omega \eta(x) \int_S h(\omega', \omega) \chi_s(\tilde{\omega}, \omega') \Phi_\lambda(t, \omega', x) d\omega' \times S_{2, \lambda}^*(t, \omega, x) dx d\omega dt,
\]

\[
J_3(\lambda, s) = \int_0^T \int_\Omega \eta(x) \int_S h(\omega', \omega) R_{1, \lambda, s}(t, \omega', x) d\omega' \times \Psi_\lambda(t, \tilde{\omega}, x) dx d\omega dt,
\]

\[
J_4(\lambda, s) = \int_0^T \int_\Omega \eta(x) \int_S h(\omega', \omega) R_{1, \lambda, s}(t, \omega', x) d\omega' \times S_{2, \lambda}^*(t, \omega, x) dx d\omega dt.
\]

and \( S_{2, \lambda}^* \) is the unique solution of

\[
\left\{
\begin{array}{l}
p_t S + \omega \cdot \nabla S - qS = -q K_{\kappa_2}^*[S] + e^{-i\lambda t} q Z_{2, \lambda}^*, \\
S(T, \omega, x) = 0, \quad (\omega, x) \in S \times \Omega, \\
S(T, \omega, \sigma) = 0, \quad (\omega, \sigma) \in \Sigma^+,
\end{array}
\right.
\]

Moreover, from (4.5) and (4.2), it follows that \( L(\lambda, r, s) \to L(\lambda, s) \), where

\[
L(\lambda, s) = \int_0^T \int_{\partial \Omega} (\tilde{\omega} \cdot \nu(\sigma))^+ (\tilde{A}_1[f_\lambda, s] - \tilde{A}_2[f_\lambda, s])(t, \tilde{\omega}, \sigma) \Psi_\lambda(t, \tilde{\omega}, \sigma) d\sigma dt
\]

\[
= \int_0^T \int_{\partial \Omega} (\tilde{\omega} \cdot \nu(\sigma))^+ (R_{1, \lambda, s}(t, \tilde{\omega}, \sigma) - R_{2, \lambda, s}(t, \tilde{\omega}, \sigma)) \Psi_\lambda(t, \tilde{\omega}, \sigma) d\sigma dt.
\]
where \( \tilde{A}_i[f, s] \) denotes the zero extension of \( A_i[f, s] \) on \( \partial \Omega \). Therefore, by taking the limit as \( r \to 1^- \) in (4.4), we have

\[
J_1(\lambda, s) + J_2(\lambda, s) + J_3(\lambda, s) + J_4(\lambda, s) = L(\lambda, s).
\]

Now, it is time to take the limit as \( s \to 1^- \). For the first two terms of the right hand side of the above identity, we get (for \( i = 1, 2 \))

\[
J_i(\lambda, s) \to J_i(\lambda), \quad J_1(\lambda) = \int_0^T \int_\Omega \eta(x) h(\bar{\omega}, \bar{\omega}) \Phi_\lambda(t, \bar{\omega}, x) \Psi_\lambda(t, \bar{\omega}, x) \, dx \, dt
\]

\[
J_2(\lambda) = \int_0^T \int_Q \eta(x) h(\bar{\omega}, \bar{\omega}) \Phi_\lambda(t, \bar{\omega}, x) \frac{S_{2, \lambda}^s(t, \omega, x)}{2} \, dx \, d\omega \, dt.
\]

On the other hand, the dependence on \( s \) in the other terms is given by \( R_{1, \lambda, s} \) and \( R_{2, \lambda, s} \), which are the solution of (4.10)

\[
\left\{ \begin{array}{l}
\partial_t R + \omega \cdot \nabla R + qR = qK_{\kappa_j} [R] + e^{i\lambda t} qZ_{j, \lambda, s}, \\
R(0, \omega, x) = 0, \quad (\omega, x) \in \mathbb{S} \times \Omega, \\
R(t, \omega, \sigma) = 0, \quad (\omega, \sigma) \in \Sigma^+.
\end{array} \right.
\]

where

\[
Z_{j, \lambda, s}(t, \omega, x) = \int_\mathbb{S} \kappa_j(x, \omega', \omega) \chi_s(\bar{\omega}, \omega') \Phi_\lambda(t, \omega', x) \, d\omega'.
\]

It is an immediate consequence of (4.2) and the Lebesgue’s Theorem that, as \( s \to 1 \), \( Z_{j, \lambda, s} \to Z_{j, \lambda} \) in \( C([0, T]; L^2(Q)) \), where

\[
Z_{j, \lambda}(t, \omega, x) = \kappa_j(x, \bar{\omega}, \omega) \Phi_\lambda(t, \bar{\omega}, x).
\]

Hence,

\[
\lim_{s \to 1^-} R_{j, \lambda, s} = S_{j, \lambda} \quad \text{in} \quad C([0, T]; L^2(Q)),
\]

where \( S_{j, \lambda} \) is the solution of

\[
\left\{ \begin{array}{l}
\partial_t S + \omega \cdot \nabla S + qS = qK_{\kappa_j} [S] + e^{i\lambda t} qZ_{j, \lambda}, \\
S(0, \omega, x) = 0, \quad (\omega, x) \in \mathbb{S} \times \Omega, \\
S(t, \omega, \sigma) = 0, \quad (\omega, \sigma) \in \Sigma^-.
\end{array} \right.
\]
and $Z_{j,\lambda}(t, \omega, x) := c_j(x)h(\tilde{\omega}, \omega)\Phi_\lambda(t, \tilde{\omega}, x)$. Therefore, $J_i(\lambda, s) \to J_i(\lambda)$, $(i = 3, 4)$ and $L(\lambda, s) \to L(\lambda)$, where

$$J_3(\lambda) := \int_0^T \int_\Omega \eta(x) \left[ \int_S h(\omega', \tilde{\omega}) S_{1,\lambda}(t, \omega', x) d\omega' \right] \Psi_\lambda(t, \tilde{\omega}, x) dx dt,$$

$$J_4(\lambda) := \int_0^T \int_\Omega \eta(x) \left[ \int_S h(\omega', \tilde{\omega}) S_{1,\lambda}(t, \omega', x) d\omega' \right] S_{2,\lambda}(t, \omega, x) dx d\omega dt.$$

(4.14)

$$L(\lambda) := \int_0^T \int_{\partial\Omega} (\tilde{\omega} \cdot \nu(\sigma))^+ \left[ S_{1,\lambda}(t, \tilde{\omega}, \sigma) - S_{2,\lambda}(t, \tilde{\omega}, \sigma) \right] \Psi_\lambda(t, \tilde{\omega}, \sigma) d\sigma dt$$

and we obtain

$$|J_1(\lambda)| \leq |J_2(\lambda)| + |J_3(\lambda)| + |J_4(\lambda)| + |L(\lambda)|, \quad (4.15)$$

where

$$|J_2(\lambda)| \leq \|\eta\|_{\infty} \|h\|_{\infty} e^{MT} \int_0^T \int_\Omega |\phi(x - t\tilde{\omega})| S_{2,\lambda}^*(t, \omega, x) dx d\omega dt,$$

$$|J_3(\lambda)| \leq \|\eta\|_{\infty} \|\phi\|_{\infty} e^{MT} \|K_h[S_{1,\lambda}]\|_{L^2(0,T;L^2(Q))},$$

$$|J_4(\lambda)| \leq \|\eta\|_{\infty} \|K_h[S_{1,\lambda}]\|_{L^2(0,T;L^2(Q))} \|S_{2,\lambda}\|_{L^2(0,T;L^2(Q))},$$

$$|L(\lambda)| \leq \|\phi\|_{\infty} e^{MT} \int_0^T \int_{\partial\Omega} (\tilde{\omega} \cdot \nu(\sigma))^+ |S_{1,\lambda}(t, \tilde{\omega}, \sigma) - S_{2,\lambda}(t, \tilde{\omega}, \sigma)| d\sigma dt. \quad (4.16)$$

Since $\phi \in C_0^\infty(\Omega, \varepsilon)$, it follows from the choice of $\varepsilon$ that the function $(t, \omega, x) \mapsto \phi(x - t\tilde{\omega})$ belongs to $H^1_0(0, T; L^2(Q))$ (as a constant function on $\omega$). Hence, we have

$$|J_2(\lambda)| \leq \|\phi\|_{\infty} e^{MT} \|K_h[S_{1,\lambda}]\|_{H^1_0(0,T;L^2(Q))} \|S_{2,\lambda}\|_{H^{-1}(0,T;L^2(Q))}.$$ 

On the other hand, from the weak convergence to zero in $L^2(0,T;L^2(Q))$ of $S_{1,\lambda}$, it follows that

$$\lim_{\lambda \to +\infty} \|K_h[S_{1,\lambda}]\|_{L^2(0,T;L^2(Q))} = 0. \quad (4.17)$$

Hence, we have from (4.15)–(4.17) and Lemma 3.3,

$$|J_1(\lambda)| = |h(\tilde{\omega}, \tilde{\omega})| \int_0^T \int_\Omega \eta(x) \phi(x - t\tilde{\omega})^2 dx dt \quad (4.18)$$

$$\leq C(\lambda) \|\eta\|_{\infty} + C_2 \int_0^T \int_{\partial\Omega} (\tilde{\omega} \cdot \nu(\sigma))^+ |S_{1,\lambda}(t, \tilde{\omega}, \sigma) - S_{2,\lambda}(t, \tilde{\omega}, \sigma)| d\sigma dt,$$

where $C(\lambda) \to 0$ as $\lambda \to +\infty$. 

Since \((\text{supp} \, \phi + s\tilde{\omega}) \cap \Omega = \emptyset\) for all \(|s| \geq T\), we have

\[
\left| \int_0^T \int_\Omega \eta(x)\phi(x-t\tilde{\omega})^2 \, dx \, dt \right| = \left| \int_{\mathbb{R}^N} \int_0^T \eta(y+s\tilde{\omega})\phi(y)^2 \, ds \, dy \right|
\]

\[
= \left| \int_{\mathbb{R}^N} \int_{-\infty}^\infty \rho(y+s\tilde{\omega})\phi(y)^2 \, ds \, dy \right|
\]

\[
\geq \int_{\mathbb{R}^N} |P_{\tilde{\omega}}\eta(y)| \phi(y)^2 \, dy
\]

and we get

\[
|\hbar(\tilde{\omega}, \tilde{\omega})| \int_{\mathbb{R}^N} |P_{\tilde{\omega}}\eta(y)| \phi(y)^2 \, dy \leq C(\lambda)\|\eta\|_\infty +
\]

\[
C_2 \int_0^T \int_{\partial \Omega} (\tilde{\omega} \cdot \nu(\sigma))^+ |S_{1,\lambda}(t, \tilde{\omega}, \sigma) - S_{2,\lambda}(t, \tilde{\omega}, \sigma)| \, d\sigma \, dt,
\]

for some constant \(C_0 > 0\). If we denote by

\[
u_{i,j}(t,\omega,\sigma) = \chi_{s}(\tilde{\omega}_j,\omega)\Phi_{\lambda}(t,\omega,x) + R_{i,\lambda,s}(t,\omega,x), \quad i = 1, 2, \quad j = 1, \ldots, k
\]

it follows from (4.2) that, as \(s \to 1^-\), \(u_{i,j} \to u^\#_{i,j}\), where

\[
u^\#_{i,j} = \delta_{\tilde{\omega}_j} \Phi_{\lambda} + S_{i,\lambda}, \quad i = 1, 2, \quad j = 1, \ldots, k
\]

and \(\delta_{\tilde{\omega}_j}\) is the spherical atomic measure concentrated on \(\tilde{\omega}_j\).

It is clear from (4.13) that \(u^\#_{1,j}(t,\omega,\sigma) = u^\#_{2,j}(t,\omega,\sigma)\), for \(\sigma \in \Sigma^{-}_{\tilde{\omega}}\) and \(j = 1, \ldots, k\). Moreover, \(u^\#_{1,j} - u^\#_{2,j} = S_{1,\lambda} - S_{2,\lambda}\). Therefore, if \(u^\#_{1,j}(t,\tilde{\omega}_j,\sigma) = u^\#_{2,j}(t,\tilde{\omega}_j,\sigma)\) on \(\Sigma^{+}_{\tilde{\omega}_j}\) for \(j = 1, \ldots, k\), it follows that

\[
C_0\|c_1 - c_2\|_\infty \leq C(\lambda)\|c_1 - c_2\|_\infty
\]

and the conclusion follows easily if we choose \(\lambda > 0\) large enough.
REFERENCES


